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ASYMPTOTIC ANALYSIS OF A CLASS OF SEPARATED FLOWS

BY

ANAND KUMAR

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DEPARTMENT OF MECHANICAL ENGINEERING

INDIAN INSTITUTE OF TECHNOLOGY KANPUR

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ASYMPTOTIC ANALYSIS OF A CLASS OF SEPARATED FLOWS

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JUNE '76

A Thesis Submitted
In Partial Fulfilment of the Requirements
for the Degree of
MASTER OF TECHNOLOGY

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CERTIFICATE

This is to certify that this work has been carried out under my supervision and has not been submitted for a degree elsewhere.

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Anand Kumar

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ABSTRACT

In this thesis, an asymptotic method has been used to analyse at large Reynolds number the flow in a sudden expansion in a channel which belongs to the class of internal separated flows.

It is argued that Navier - Stokes equations in a limit appropriate to this class reduce to boundary layer equations, although effect of viscosity is not confined to thin layers and reversed flow is present. Expansions valid far away from the sudden expansion are compatible with results obtained by other workers who have used complete Navier - Stokes equations for the flow developing into a Poiseuille flow. Coordinate expansions near the sudden expansion call for three different limits valid in the three flow regions, namely, core, mixing layer and bubble. A matching principle is invoked for these coordinate expansions which is similar to the one used for parameter expansions.

Study of the initial development of the mixing layer has led to the identification of a new class of viscous layers called weak mixing layers across which sharp change in vorticity occurs but not velocity. A family of similar solutions is given and their properties have been discussed.

1. INTRODUCTION

Flow separation is a common engineering phenomenon. It may increase or decrease the usefulness of an engineering device. Many fluidic devices are based on flow separation while the stall of an airfoil at large angle of incidence results in large drag and loss of lift. Flow separation can be caused by an adverse pressure gradient which retards the fluid, the effect being more pronounced on the low momentum fluid near the wall. This fluid ultimately unable to overcome the opposing pressure deflects into the stream causing a reversal of flow immediately downstream. This is termed 'boundary layer separation'. Perhaps the best known example of such behaviour is the flow separation from the rear of a circular cylinder placed transverse to a stream of fluid.

In engineering practice a common cause of flow separation is an abrupt change in the profile of a solid surface in contact with the fluid. In an unconfined flow it may be caused by a depression in, or a protrusion from, an otherwise smooth surface, such as a step in an open channel. An example in confined flow is the change in diameter of a pipe or the change in width of a channel.

Investigations on flow separation have proceeded in several different directions. One class has dealt with the question of whether a boundary layer separates under a given pressure distribution and if so where is the separation point. Another class deals with the inviscid analysis of a separated flow past a body in which the separating streamline is specified by some geometric or dynamic condition (Free streamline theories). This class predicts drag as a function of a base pressure coefficient. Still another class deals with flows in cavities. This class has the advantage that the length seales in the two directions are determined by the geometry of the solid boundaries and the behaviour at large Reynolds number is such that a core region of uniform vorticity develops. Batchelor (1956) first pointed out this behaviour and it has received support from direct numerical computations and experimental data.

Although the number of investigations of separated flows is extremely large, it cannot be said that a definitive basis has been laid down. (Roshko). Even direct numerical integration of Navier-Stokes equations encounters difficulty as Reynolds number becomes large (e.g. Hung and Macagno 1966). In the recent past there has been a trend to use the asymptotic method to handle the flow situations which otherwise looked to be intractable. This investigation is taken up with a similar intention.

There are many facets of a separated flow problem. The manner in which the asymptotic method can be used depends on the class of problem considered. This thesis deals with the separated flow in a channel having sudden expansion when Reynolds number is large. This problem is choosen because the task of predicting flow separation which in itself is a major concern, is reduced. It is also thought to be a member of a class of internal separated flows where length scales in two directions are not determined by the geometry of the boundary. Also, numerical results of Hung and Macagno (1966) can be used to check the appropriateness of the selected expansions.

When separation occurs in pipe or channel system, or in rotor, stator or casing of pumps, turbines or compressors, the resulting flow is a member of the class internal separated flow. There are two major features of this class. Often vorticity has diffused over the most of the fluid, unlike what happens in external flow. Also continuity and confinement of the flow by the wall produces an interaction between the region close to the wall and region away from the wall and this interaction strongly influences pressure distribution. In external flows the pressure distribution is by and large determined by the flow away from the wall.

The mathematical problem will be to solve the governing equation which is elliptic in nature, with prescribed condition on a section upstream of the expansion and on a section

downstream. Certain conditions such as fully developed flow at both the sections are used and these may be taken to describe the actual conditions approximately. But then the task is not simple as the theory of such partial differential equations is in a far from satisfactory state. Instead the use is made of the asymptotic theory by taking a limit in which Reynold number approaches infinity. This limit is mainly designed to take fullest advantage of large Reynolds numbers often found in practice. It provides a systematic method of constructing successively more refined approximations. Also it points out the dominant forces in different regions.

Based on the computation performed by Hung and Macagno (1966), a simplification is possible. It is argued in an all together a different way that the governing equation (for the most of the flow regions) are like boundary layer equations. This perhaps supports the intutive feeling that viscous forces and inertia forces are important in the lowest order approximation and mostly the information will be conveyed in the downstream direction. This approach and other aspects of formulation of the problem and outline of the method are discussed in Chapter 2.

A new family of viscous layers termed weak mixing layer were identified at early stages of the present investigation. Weak mixing layers are viscous layers across which sharp changes in vorticity takes place but not in

velocity. Chapter 3 deals with weak mixing layer and is complete in itself.

Chapters 4 and 5 contain the details of expansions for large and small X and the last chapter deals with conclusions.

2. FORMULATION OF THE PROBLEM

2.1 ASSUMPTIONS OF THE PROBLEM :

Flow through a sudden expansion is considered. The flow is assumed to be steady, two-dimensional laminar flow of an incompressible Newtonion fluid of constant properties.

The half-width of the channel after the expansion and the average velocity after the expansion is used as length and velocity scales for non-dimensionalising. The governing equations then are

$$u_x + v_y = 0$$

$$uu_x + vu_y = - p_x + R^{-1} (u_{xx} + u_{yy}) \qquad (2.1)$$

$$uv_x + vv_y = - p_y + R^{-1} (v_{xx} + v_{yy})$$

where x is the (dimensionless) downstream co-ordinate and y is the (dimensionless) transverse co-ordinate, the origin being at the expansion in the centre of channel. u and v are the non-dimensionalised velocities in x- and y-directions respectively and p is the non-dimensionalised pressure. R is the Reynold number based on half channel width and average velocity after the expansion. The suffixes denote partial differentiation.

The half-width of the channel before the expansion is denoted by a.

The boundary conditions are

$$x = 0 \;:\; u = (3/2a)\left[1-(y/a)^2\right] \qquad 0 \leq |y/a| \leq 1$$
$$= 0 \qquad 0 < a \leq |y| \leq 1 \qquad (2.2a)$$
$$v = 0 \qquad 0 \leq |y| \leq 1$$

$$x \to \infty \;:\; u \to (3/2)(1-y^2) \qquad 0 \leq |y| \leq 1$$
$$v \to 0 \qquad 0 \leq |y| \leq 1 \qquad (2.2b)$$

$$y = \pm 1 \;:\; u = v = 0 \qquad 0 \leq x < \infty \qquad (2.2c)$$

The flow is further assumed to be symmetrical about x-axis.

$$y = 0 \;:\; u_y = 0,\; v = 0 \qquad 0 \leq x < \infty \qquad (2.2d)$$

Only the upper half of the channel need be considered in the analysis on account of symmetry.

2.2 OUTLINE OF THE FLOW REGIONS :

Now we consider the flow at large Reynold number. It is felt that the flow may be devided into several regions based on some properties characteristic of the regions. Then each region can be handled separately and later matching in a limiting sense can be performed.

After the sudden expansion three different regions can be identified, the main flow, the bubble and the mixing layer. The main flow forms the core and expands to fill

the whole width of the channel. The bubble also known as dead fluid region as it has the slow recirculating fluid entrapped by the main flow. The bubble plays a passive role in forming the main flow with a rather small energy exchange. (Hung & Macagno 1966). The region between the bubble and the main flow is termed a mixing layer or a shear layer.

2.3 LARGE REYNOLDS NUMBER LIMIT :

The core flow which separates after the sudden expansion reattaches downstream. For large Reynold number, the length of reattachment behaves like R as is known from the work of Hung & Macagno. The length of reattachment can thus be taken to represent the length scale in x-direction. The length scale in y-direction is taken as one.

Another basis for this selection of length scales can also be given. Let the length scales in x- and y-directions be L and unity respectively. Then the inertia terms in equation (2.1) are of the order of $1/L$, since the velocity is clearly of the order unity. The viscous terms are of the order of $1/R$. Before the expansion the viscous terms were significant in the flow. Hence one may suppose that inertia and viscous terms are of the same order in the core. Hence L is of the order of R.

The appropriate length scales in x- and y-directions for the bubble are the length and height of the bubble

respectively. Hence the bubble has the same length scales as the core. Since the mass flow is small in the mixing layer, the net mass flow in the bubble is of the same order as in the core. Hence the velocity scales are also the same.

Since the mixing layer has a velocity field induced by the core and since it entrains the fluid from the bubble, its velocity scales are the same as the core and the bubble. Further its length scale in x-direction is clearly the same as the core. It then follows from the continuity that the length scale in y-direction also must be the same as the core.

We therefore introduce the new variables

$$\begin{aligned} X &= x/R \quad , \quad Y = y \\ U &= u \quad , \quad V = Rv \end{aligned} \tag{2.3}$$

The following equations are then obtained

$$\begin{aligned} &U_X + V_Y = 0 \\ &UU_X + VU_Y = - p_X + U_{YY} + R^{-2} U_{XX} \\ &p_Y + R^{-2} (UV_X + VV_Y - V_{YY}) = R^{-4} V_{XX} \end{aligned} \tag{2.4}$$

We now consider the limit $R \to \infty$ for fixed X and Y. The assumption based on the physical reasoning of the earlier paragraphs is that the limit $R \to \infty$ for fixed X,Y describes the behaviour in all the three regions. The limiting equations

are

$$U_X + V_Y = 0$$

$$UU_X + VU_Y = - p_X + U_{YY} \qquad (2.5)$$

$$p_Y = 0$$

These are similar to the well known boundary layer equations although the length scales in the boundary layer are different and corresponding limit is also different.

There is also another difference between a classical boundary layer problem and the present problem. In the former, there are two velocity conditions to be satisfied at the solid wall and one at the outer edge of the layer. Here there are two velocity conditions at the two walls. Thus there are three boundary conditions in the classical boundary layer problem and four in the present case. The difference can be accounted readily by noting that the pressure gradient is known in the boundary layer. Here it is not known. If one eliminates pressure, the order of the differential equation increases by one thus accounting for additional boundary condition.

With the streamfunction ψ given by

$$U = \psi_Y \, , \quad V = - \psi_X \qquad (2.6)$$

equation (2.5) becomes

$$\psi_{YYY} + \psi_X \psi_{YY} - \psi_Y \psi_{XY} = P_X(X) \qquad (2.7)$$

where P_X (X) is the pressure gradient which depends on X only. The boundary conditions in terms of ψ are

$$X = 0 \;:\; \psi = (3/2)\left[(Y/a) - (Y/a)^3/3\right] \qquad 0 \leq |Y/a| \leq 1 \qquad 2.8a$$
$$= 1 \qquad 0 < a \leq |Y| \leq 1$$

$$Y = \pm 1 : \psi = \pm 1 \;,\; \psi_Y = 0 \qquad 0 \leq X < \infty \qquad 2.8b$$

$$Y = 0 \;:\; \psi = 0 \;,\; \psi_{YY} = 0 \qquad 0 \leq X < \infty \qquad 2.8c$$

The flow approaches a fully developed velocity profile

$$X \to \infty \;:\; \psi \to 3/2\,(Y - Y^3/3) \qquad 0 \leq |Y| \leq 1 \qquad 2.8d$$

2.4 OUTLINE OF METHOD OF CONSTRUCTION OF SOLUTIONS :

Now we proceed to construct analytical solutions valid in different parts of the flow. The decision to use analytical solutions was based on several considerations. It was anticipated that as the initial profile is not analytic, some singularities will result and the usual numerical schemes would not be reliable. Also, since usual numerical schemes are not designed for the reversed flow, some major modifications would be required. The results in analytical form would be valid for any a, while numerical result would be valid for specific values of a.

The outline of the method is to construct solutions for large X and also for small X and to match or patch at a

suitable point.

The construction of solution for small X turns out to be very complex and new ideas had to be developed to meet the complexity. Three different series solutions each valid in core, or mixing layer, or bubble are required. They have to be matched so that they are mutually compatible. Although matching of parameter expansions is wellknown this is probably the first example where matching of coordinate expansions is required. This is taken up in Chapter 5.

For large X the flow may be represented by an asymptotic expansion whose gauge functions depend on X, such that as $X \to \infty$ the flow attains the fully developed velocity distribution. The treatment for large X has been done in Chapter 4.

Investigation of early stages of the shear layer led to a new family of viscous layers which may be termed as weak mixing layers. Weak mixing layers are viscous layers across which sharp changes in vorticity takes place but not in velocity. As the subproblem of weak mixing layers provides some information about the expansions to be used later, it is taken up first in Chapter 3 which is complete in itself.

3. WEAK MIXING LAYERS

3.1 INTRODUCTION :

Thin layers across which large changes in velocity occur and which are distant from solid walls have been extensively studied. (Rosenhead 1963, Lock 1951, Görtler 1942). However, somewhat similar layers across which large changes take place in vorticity but not in velocity do not seem to have been investigated. Mixing layers of these two types may be called as strong and weak mixing layers.

While momentum diffusion is the main feature of the strong mixing layers, diffusion of vorticity is the primary feature of the weak mixing layer. The mechanism of diffusion may be the motions at molecular scale which are described phenomenalogically by viscosity or it may be large scale eddying motions found in turbulent flows.

The contents of this chapter throw light on the type of expansions to be used in the mixing layer of the flow through sudden expansion. Some information which is not pertinent to the flow through sudden expansion is also added for the sake of completeness.

The analytical study in the next sections deals with the mixing layer between two parallel streams having constant, but different, vorticity. Boundary layer approximations are used and attention is focussed on mixing layers having similar profiles. If turbulence is present and if its effects can be approximately described by a suitable eddy viscosity, analysis can be carried out essentially along the same lines as the analysis in the laminar case.

3.2 LAMINAR WEAK MIXING LAYER :

Consider the steady two dimensional flow of a Newtonian fluid of constant properties. The boundary layer approximations are assumed to be valid and pressure is taken to be uniform as in the case of strong mixing layer. The flow is then governed by the equations.

$$u_x + v_y = 0 \tag{3.1a}$$

$$uu_x + vu_y = \nu\, u_{yy} \tag{3.1b}$$

and it is subjected to the conditions

$$x = 0 \;:\; y \geq 0 \qquad u = \Omega_1 y \tag{3.2a}$$

$$y \leq 0 \qquad u = \Omega_2 y \tag{3.2b}$$

$$x > 0 \;:\; y \to +\infty \quad u_y \to \Omega_1 \tag{3.2c}$$

$$y \to -\infty \quad u_y \to \Omega_2 \tag{3.2d}$$

Here x and y are Cartesian co-ordinates in streamwise and normal direction, u and v are components of velocity, ν is

kinematic viscosity and Ω_1 and Ω_2 are imposed vorticity by upper and lower streams. There is a discontinuity in the vorticity profile at the initial section ($\Omega_1 \neq \Omega_2$). In absence of reversed flow at the initial section, $\Omega_1 \geq 0$ and $\Omega_2 \leq 0$.

The last two conditions arise from the conservation of vorticity outside the weak mixing layer.

If similarity solution exists with length and velocity scales behaving like x^p and x^q, (3.1b) requires that $2p + q$ is one. Conditions (3.2c) and (3.2d) further require that p is equal to q. Thus the length and velocity scales behave like $x^{1/3}$.

Let the stream function ψ be given by

$$\begin{aligned} \psi &= \Omega_1 (3\nu x/2\Omega_1)^{2/3}\,\bar{f}(\bar{\eta}), \\ \bar{\eta} &= (3\nu x/2\Omega_1)^{-1/3}\,y \end{aligned} \tag{3.3}$$

Then the equation of motion becomes

$$\bar{f}''' + \bar{f}\,\bar{f}'' - 1/2\,\bar{f}'^2 = 0 \tag{3.4a}$$

where prime denotes differentiation. The conditions (3.2c) and (3.2d) require that

$$\bar{\eta} \to \infty : \bar{f}'' \to 1 \;, \quad \bar{\eta} \to -\infty : \bar{f}'' \to \Omega_2/\Omega_1 \tag{3.4b}$$

The third boundary condition required for the third order equation (3.4a) locates the layer. There are several options for the third boundary condition. Two simple alternatives are

$$\bar{f}\,(0) = 0 \tag{3.5a}$$

$$\bar{f}''\,(0) = 1/2\,(1+\Omega_2/\Omega_1) \tag{3.5b}$$

If $\bar{f}\,(\bar{\eta})$ is a solution of equation (3.4a) then $a\bar{f}\,(a\bar{\eta}+b)$ also satisfies equation (3.4a) for any constants a and b. Since one solution $\bar{f}\,(\bar{\eta})$ satisfying equation (3.4) can generate a family of solutions $\bar{f}\,(\bar{\eta}+b)$, the third boundary condition selects one member of the family. Also the above transformation is of some use in direct numerical computation.

The weak mixing layers having a similar profile are essentially governed by one parameter describing the ratio of vorticities on the edges of the layer, as shown by equation (3.4). The parameter enters in the boundary condition but not in the equation. It is possible to reformulate the problem so that the parameter occurs in the equation and not in the boundary condition. Let

$$f\,(\eta) = \left[2\Omega_1/(\Omega_1-\Omega_2)\right]^{1/3}\bar{f}(\bar{\eta})-1/2\,\lambda\eta^2 \tag{3.6a}$$

$$\eta = \left[2\Omega_1/(\Omega_1-\Omega_2)\right]^{-1/3}\bar{\eta} \tag{3.6b}$$

$$\tilde{\lambda} = (\Omega_1+\Omega_2)/(\Omega_1-\Omega_2) \tag{3.6c}$$

Then the equations (3.4a), (3.4b), (3.5a) and (3.5b) becomes

$$f''' + (f + 1/2\, \tilde{\lambda} \eta^2)(f'' + \tilde{\lambda}) - 1/2\,(f' + \tilde{\lambda}\eta)^2 = 0 \quad (3.7a)$$

$$\eta \rightarrow \pm \infty : f'' \rightarrow \pm 1 \quad (3.7b)$$

$$f(0) = 0 \quad (3.8a)$$

$$f'(0) = 0 \quad (3.8b)$$

The parameter $\tilde{\lambda}/2$ is the ratio of average vorticity of the two streams and the vorticity change across the mixing layer. $f''(\eta)$ represents in stretched co-ordinates the departure of local vorticity from its average value.

In the above formulation Ω_1 was assumed to be greater than Ω_2. Otherwise a slight modification is required in (3.6), but the problem is essentially the same as given by (3.7).

An analytical method of solution of (3.7) is to find solutions valid for large positive η, large negative η and small η and patch up these solutions at suitable intermediate points. These solutions are

$$f \sim (\eta + D)^2/2 + \exp\left\{-(\tilde{\lambda} + 1)(\eta + D)^3/6\right\} \times \sum_{m=0}^{\infty} A_m (\eta + D)^{-3m-6}, \quad \eta \rightarrow \infty \quad (3.9a)$$

$$f \sim -(\eta + E)^2/2 + \exp\left\{-(\tilde{\lambda} - 1)(\eta + E)^3/6\right\} \times \sum_{m=0}^{\infty} B_m (\eta + E)^{-3m-6}, \quad \eta \rightarrow -\infty \quad (3.9b)$$

$$f \sim \sum_{m=0}^{\infty} C_m \eta^m \quad , \eta \rightarrow 0 \tag{3.9c}$$

The following are the recursive relations for the constants A_m , B_m and C_m .

$$A_m = -4 (\tilde{\lambda} + 1)^{-2} \Big[3 (m+1) (\tilde{\lambda} + 1) A_{m-1} + (3m+1) (3m+2) A_{m-2} \Big], \text{ for } m > 0, A_m = 0 \text{ for } m < 0 \tag{3.10a}$$

$$B_m = -4 (\tilde{\lambda} - 1)^{-2} \Big[3 (m + 1) (\tilde{\lambda} - 1) B_{m-1} - (3m + 1) (3m + 2) B_{m-2} \Big] \text{ , for } m > 0 \text{ ,} \tag{3.10b}$$

$$B_m = 0 \text{ for } m < 0$$

$$C_{m+3} = \Big[(m + 1) (m + 2) (m + 3) \Big]^{-1} x \Big[-\tilde{\lambda} C_m / 2 \; x \; (m - 1) (m-2) + \sum_{n=0}^{m} \Big\{ 1/2 \, (n + 1) (m - n + 1) x \, C_{n+1} C_{m-n+1} - (n + 1) (n + 2) C_{n+2} C_{m-n} \Big\} \Big] \text{ ,for } m > 0 \tag{3.10c}$$

If the magnitude of vorticity of the upper stream is larger than that of lower stream and if the reversed flow is absent, $\tilde{\lambda}$ lies in the range of (0,1). The above solution is applicable in this range except when $\tilde{\lambda} = 1$. This case arises when a vortical stream comes in contact with stationary fluid. Then (3.9b) and (3.10b) are replaced by

$$f \sim \sum_{m=0}^{\infty} B_m \exp (B_0 \, m \, \eta) \text{ ,} \quad \eta \rightarrow -\infty \tag{3.9d}$$

$$B_m = -\left[B_0\, m^2\,(m-1)\right]^{-1} \sum_{n=1}^{m-1} (1/2)\, n\,(m-3n) B_n\, B_{m-n}$$

$$\text{for } m > 1 \qquad (3.10d)$$

The recursive relations express all the coefficients A_m, B_m and C_m in terms of lower coefficients and in all seven coefficients, namely, A_0, B_0, C_0, C_1, C_2, D and E (or B_1 in case $\tilde{\lambda} = 1$) are left to be specified. One of these coefficients is determined from the arbitrarily chosen third boundary condition (3.8a) or (3.8b). The remaining six coefficients can be determined by patching at two suitable points. One patching condition is that f and its derivatives are continuous. An alternate condition would be to minimise a positive definite measure of the jumps in f and its first two derivatives.

3.3 TURBULENT WEAK MIXING LAYER :

Consider stationary turbulent flow. It is assumed that if the length and velocity scales of mean motion behave like Mx^p and Nx^q, the effective kinematic viscosity behaves like $KMNx^{p+q}$, as a first approximation. The constant K is independent of boundary and initial conditions. The equations of mean motion and continuity and boundary conditions are similar to (3.1) and (3.2). The index p is found to be one from equation (3.1). As in the laminar case, the condition (3.2c) and (3.2d) require that p and q are equal. Let the mean stream function ψ be given by

$$\psi = MNx^2\, F(\bar{\eta}) \quad , \quad \bar{\eta} = y/Mx \qquad (3.11a)$$

and let

$$\nu_e = KMNx^2 \tag{3.11b}$$

The equation of mean motion reduces to (3.4a) and boundary conditions to (3.4b), if M and N are taken to be K/2 and $\Omega_1 K/2$. Thus the similar profile of turbulent weak mixing layer is governed by the same equation as its laminar counterpart.

3.4 RESULTS AND DISCUSSION :

The results given in Table 1 were obtained by direct numerical integration of (3.4a) with boundary conditions (3.4b) and (3.5a) and have been shown in Figures 1 to 3. This three point boundary value problem over doubly infinite range was solved by iterative scheme. $\bar{f}(0)$, $\bar{f}'(0)$ and $\bar{f}''(0)$ were assumed and integration was carried out for positive and negative $\bar{\eta}$. The initial values $\bar{f}'(0)$ and $\bar{f}''(0)$ were adjusted till the boundary conditions (3.4b) were satisfied. Calculations were performed on IBM7044 of the Indian Institute of Technology, Kanpur using fourth order Runge-Kutta method with Gills variation. Steps of $\Delta\bar{\eta} = 0.01$ were found to be satisfactory. A check with a step of $\Delta\bar{\eta} = 0.005$ for $\Omega_2/\Omega_1 = 0,-1$ showed that the fourth digit in vorticity was not affected by the step size.

The growth of weak and strong mixing layers is quite different when the flow is laminar. The thickness of the

layers behaves like $x^{1/3}$ and $x^{1/2}$ respectively. However the growth of layers of both type is linear when the flow is turbulent.

Figure 1 shows the velocity profile which can be described as a rounded corner profile. Figure 2 shows the variation of vorticity (to the boundary layer approximation). In the central portion the curve is fairly straight. The change of slope is quite rapid at the upper edge and relatively less rapid at the lower edge. As the vorticity ratio $|\Omega_2/\Omega_1|$ diminishes the profile developes a long tail towards the lower edge and the thickness of weak mixing layer increases.

Figure 3 shows the distribution of normal velocity. It has a linearly increasing behaviour at the upper and lower edge of the mixing layer when $\Omega_2/\Omega_1 \neq 0$. This behaviour is different from that in strong mixing layer. To see how such a behaviour arises, suppose that the normal velocity approaches a limit for large positive $\bar{\eta}$ and also for large negative $\bar{\eta}$ or increases less rapidly than $\bar{\eta}$. Any streamline sufficiently far away from the mixing layer then has zero slope as v/u approaches zero. Now consider a control volume bounded by one such streamline at the upper edge and one at the lower edge, and by two sections $x = x_1$ and $x = x_2$. Since the flow accelerates in the downstream direction, and since the areas at the upstream and downstream sections are equal, the mass efflux through the downstream section will exceed the mass

influx through the upstream section. This violation of continuity shows that the normal velocity cannot approach a limiting value as $\bar{\eta} \to \infty$ and as $\bar{\eta} \to -\infty$. The above argument cannot be applied to the case when the fluid on the lower edge is stationary ($\Omega_2/\Omega_1 = 0$).

If the third boundary condition (3.5) is chosen in a different way, the shape of the u velocity profile remains unaltered and it merely gets displaced. However, the shape of the normal velocity profile changes, as the normal velocity changes from $2\bar{f}-\bar{\eta}\bar{f}'$ to $2\bar{f}-\bar{\eta}\bar{f}'-b\bar{f}'$ under the transformation $\bar{f}(\bar{\eta})$ to $\bar{f}(\bar{\eta}+b)$. Consequently, for one particular choice of boundary condition (3.5), one can obtain a normal velocity pprofile in which v approaches a limiting value at the upper or the lower edge. The previous argument however shows that the normal velocity cannot be made to approach a limit for $\bar{\eta} \to \infty$ and at the same time for $\bar{\eta} \to -\infty$.

The weak mixing layers for which Ω_2/Ω_1 is positive are qualitatively different from those studied here. The initial profile in this case has reversed flow and hence the convective influences are carried in the negative x-direction. This upstream propagation would also render the specification of initial condition with a sharp corner unrealistic.

A generalisation of the initial condition (3.2a) and (3.2b) would be

$$x = 0 \ : \ y \geqslant 0 \qquad u = U_0 + \Omega_1 y \tag{3.12a}$$

$$y \leqslant 0 \qquad u = U_0 + \Omega_2 y \tag{3.12b}$$

The problem is however somewhat different when U_0 is not zero. In the previous case, as $\nu \to 0$, velocity in the viscous layer was $o(1)$ and in the second case it is $O(1)$. If one takes

$$u = U_0 + \epsilon u_1 \quad , \qquad v = \epsilon \delta v_1 \tag{3.13}$$

the governing equation becomes

$$U_0 u_{1x} = \nu u_{1yy} \tag{3.14}$$

where the transverse length scale δ is taken to be $\nu^{1/2}$.

4. EXPANSION FOR LARGE X

It is assumed that at downstream far from the sudden expansion the velocity distribution tends to fully developed flow. The expansion for stream function for large X can be written as

$$\psi = \sum_n g_n(X)\ \psi_n(Y) \qquad (4.1)$$

where $g_{n+1}/g_n \to 0$ as $X \to \infty$ and $g_0 = 1$. ψ_0 represents the stream function corresponding to fully developed flow.

$$\psi_0 = (\ 3Y - Y^3\ )/2 \qquad (4.2)$$

Substitution of (4.1) into (2.7) yields

$$\sum_n \Big[g_n\ \psi_n''' + \sum_{r+s=n} g_r'\ g_s\ (\ \psi_r\ \psi_s'' - \psi_r'\ \psi_s'\ )\Big] = P_X\ (X) \qquad (4.3)$$

If we assume that $g_r'\ g_s = -\lambda_{rs}\ g_{r+s}$, then $g_r' = -\lambda_{r0}\ g_r$ as $g_0 = 1$. Hence $g = \exp(-\lambda_{r0} X)$. Also $-\lambda_{r0}\, g_r\, g_s = -\lambda_{rs}\, g_{r+s}$. This can be satisfied by taking $g_r = \exp(-r\lambda X)$. With this choice of g_r, equation (4.3) reduces to

$$\psi_n''' + n\lambda\big[(3/2)(1-Y^2)\psi_n' + 3Y\psi_n\big] = P_n + \lambda \sum r\left(\psi_r \psi_s'' - \psi_r' \psi_s'\right) \qquad (4.4)$$

for all n. $P_X(X)$ is assumed to be $\sum_n g_n P_n$ where P_n's are unspecified. Eleminating P_n s from (4.4) leads to

$$\psi_n^{iv} + n\lambda\left[(3/2)(1-Y^2)\psi_n'' + 3\psi_n\right] = \lambda \sum_{\substack{r+s=n \\ s>0}} r\left(\psi_r\psi_s''' - \psi_r''\psi_s'\right) \qquad (4.5a)$$

The boundary conditions on ψ imply that

$$Y = \pm 1 \; : \quad \psi_n = \psi_n' = 0 \quad , n \neq 0 \qquad (4.5b)$$

For the case when n is equal to one (4.5a, b) is

$$\psi_1^{iv} + \lambda\left[(3/2)(1-Y^2)\psi_1'' + 3\psi_1\right] = 0 \qquad (4.6a)$$

$$Y = \pm 1 \; : \quad \psi_1 = \psi_1' = 0 \qquad (4.6b)$$

where λ appears as eigenvalue.

A simple method of determining λ is to seek a series solution of the form

$$\psi_1 = \sum_n \alpha_n Y^n \qquad (4.7a)$$

where the coefficients α_n depend on λ. The recursive relation for the coefficients is obtained from equation (4.6a) as

$$\alpha_{n+4} = -3/2\,\lambda\,\frac{(n+2)\,\alpha_{n+2} - (n-2)\,\alpha_n}{(n+4)\,(n+3)\,(n+2)} \qquad (4.7b)$$

for all n.

A general solution of (4.6a) can be written as

$$\psi_1 = \beta_1\psi_{11} + \beta_2\psi_{12} + \beta_3\psi_{13} + \beta_4\psi_{14} \qquad (4.8)$$

where ψ_{1i}'s are linearly independent functions of Y

satisfying (4.6a) and β_i s are constants to be evaluated.

The functions ψ_{1i} are obtained from equation (4.7a, b) on taking $\alpha_0 = 0$, $\alpha_1 = 1$, $\alpha_2 = \alpha_3 = 0$ we get

$$\psi_{11} = Y - \lambda/40\, Y^5 + \lambda^2/1120\, Y^7 - \ldots \tag{4.9a}$$

taking $\alpha_0 = \alpha_1 = \alpha_2 = 0$, $\alpha_3 = 1$

$$\psi_{12} = Y^3 - 3\lambda/40\, Y^5 + \lambda/140\,(1+3\lambda/8)\, Y^7 - . \tag{4.9b}$$

taking $\alpha_0 = 1$, $\alpha_1 = \alpha_2 = \alpha_3 = 0$

$$\psi_{13} = 1 - \lambda/8\, Y^4 + \lambda^2/160\, Y^6 - \ldots \tag{4.9c}$$

and taking $\alpha_0 = 1$, $\alpha_1 = 0$, $\alpha_2 = -1$, $\alpha_3 = 0$

$$\psi_{14} = 1 - Y^2 \tag{4.9d}$$

While obtaining the above solutions the coefficients α_0, α_1, α_2 and α_3 are chosen in a way such that the odd and even solutions do not get mixed. ψ_{11} and ψ_{12} are odd functions of Y whereas ψ_{13} and ψ_{14} are even.

For a nontrivial solution condition (4.6b) would require that

$$\begin{vmatrix} \psi_{11}(1) & \psi_{12}(1) & \psi_{13}(1) & \psi_{14}(1) \\ \psi_{11}'(1) & \psi_{12}'(1) & \psi_{13}'(1) & \psi_{14}'(1) \\ \psi_{11}(-1) & \psi_{12}(-1) & \psi_{13}(-1) & \psi_{4}(-1) \\ \psi_{11}'(-1) & \psi_{12}'(-1) & \psi_{13}'(-1) & \psi_{14}'(-1) \end{vmatrix} = 0$$

which on simplification yields

$$\left\{ \begin{vmatrix} \psi_{11}(+1) & \psi_{12}(+1) \\ \psi_{11}'(+1) & \psi_{12}'(+1) \end{vmatrix} \right\} \times \left\{ \psi_{13}(1) \right\} = 0 \tag{4.10}$$

$$\text{or } Q(\lambda) = Q_1(\lambda) \times Q_2(\lambda) = 0 \tag{4.11}$$

where Q_1 and Q_2 are first and second curly brackets in equation (4.10) respectively.

The zeros of $Q(\lambda)$ are the eigenvalue of (4.6). $Q(\lambda)$ in (4.11) has been represented as product of $Q_1(\lambda)$ and $Q_2(\lambda)$. The zeros of $Q_1(\lambda)$ are the eigenvalue corresponding to odd eigenfunction and that of $Q_2(\lambda)$ to even eigenfunction. A simple computation yields the eigenvalue.(Table 2A).

Gillis and Brandt (1964) attempted a direct numerical solution of the equation obtained from the complete Navier-Stokes equation. They are concerned with development of Poiseuille flow. The relevant results are contained in Table 2B. Gillis and Brandt are concerned only with odd eigenfunction and find that there is a critical Reynold number above which the eigenvalue is real. ($R_{crit.}$ = 8.46 approximately).

Wilson (1969) again starting from complete Navier-Stokes equation finds that for large R there are two sequences of eigenvalues. Both sequences are asymptotically

real as $R \to \infty$. The member of the first sequence are $O(R)$ as $R \to \infty$. The second sequence consists of eigenvalues which are $O(1)$ as $R \to \infty$. The flow will be dominated by the member of the second sequence for large R and the disturbance will persist downstream a distance $O(1)$. Wilson used a expansion in powers of R^{-1} for the eigenvalue.

$$\lambda = \sum_n (R^{-1})^n \bar{\lambda}_n \tag{4.12}$$

The results are given in Table 2C. Comparison of the present results with those of Wilson and Gillis and Brandt shows that the limit considered in Chapter 2 and the expansion (4.1) used here provide excellent approximations.

Interestingly the dominant mode of disturbance velocity is antisymmetrical. It follows as this will have fewer zeros than symmetrical velocity perturbation. Because of the assumed symmetry of flow in the problem of sudden expansion the eigenvalues of our interest are odd. Therefore we get

$$\begin{aligned} &\beta_3 = \beta_4 = 0 \\ &\psi_1 = \beta_1\left(\psi_{11} + (\beta_2/\beta_1)\psi_{12}\right) \end{aligned} \tag{4.13}$$

where $\beta_2/\beta_1 = -\psi_{11}(1)/\psi_{12}(1) = -\psi_{11}'(1)/\psi_{12}'(1)$

The magnitude of β_1 has to be determined by patching.

The problem of finding higher ψ_n is simple. For $n \geq 2$ the right hand side of (4.5a) is a forcing function which is known.

The gauge function were chosen on the basis of the assumption $g_r' \, g_s = -\lambda_{rs} \quad g_{r+s}$. It is possible to generalise this expansion in the following way

$$\psi = \psi_o + \sum_{m,n \geq 1} \exp(-n\lambda_m X) \; \psi_{mn}(Y) \tag{4.14}$$

where the double series is suitably arranged so that the gauge functions are successively of higher order, and $\lambda_m < \lambda_{m+1}$. λ_m are the eigenvalue of

$$\psi_{m1}^{iv} + \lambda_m \left[(3/2)(1-Y^2)\, \psi_{m1}'' + 3\,\psi_{m1} \right] = 0 \tag{4.15a}$$

$$Y = \pm 1 \; : \; \psi_{m1} = \psi_{m1}' = 0 \tag{4.15b}$$

The eigenvalues are numbered starting from one. The problem of finding ψ_{mn}, $n > 1$ is similar to what has been discussed earlier.

The eigenvalue with negative real part may also be used. (Wilson 1969). For large R the only eigenvalues whose real part is negative are those O(R). It can therefore be said that the disturbance upstream of the sudden expansion penetrates a distance $O(R^{-1})$. (Since the relevant Reynold number for upstream channel is like the Reynold number for downstream flow).

5. EXPANSION FOR SMALL X

5.1 INTRODUCTION :

In this chapter solutions for various regions as outlined in section 2.4 are constructed and matching is considered in the limit $X \to 0$. A general methodology to construct the series solution would be to write the independent variables in the form

$$\xi = (mX)^{1/m}, \quad \eta = [Y - \Delta(\xi)]/\delta(\xi) \tag{5.1a}$$

where $\Delta(\xi) = \sum_n \Delta_n \xi^n$, $\delta(\xi) = \sum_n \delta_n \xi^n$. The variables are so choosen to take care of fractional powers of X by taking m appropriately. The definition of η means that Y is stretched about the curve $\Delta(\xi)$, the stretching function being $\delta(\xi)$.

The form of stream function is choosen as

$$\psi = \sum_n \xi^{k+n} [\log \xi . g_n(\eta) + f_n(\eta)] \tag{5.1b}$$

It is expected that log term will also appear in some region at somewhat later stage and so a general form of stream function is written. g_n shall be taken zero unless required. By specifying k, the leading term in the expansion of stream function can be adjusted. Substitution of (5.1) in (2.7) leads to

$$f_n''' + \sum_{k+r+s+l=n+m} \delta_l \{ f_s'' [(k+r) f_r + g_r] - f_r' [(k+s-l) f_s' + g_s'] \} = \text{Constant.} \quad (5.2a)$$

$$g_n''' + \sum_{k+r+s+l=n+m} \delta_l \{ g_s'' [(k+r) f_r + g_r] + f_s'' g_r (k+r) - g_r' [(k+s-l) f_s' + g_s'] - (k+s-l) f_r' g_s' \} = \text{Constant.} \quad (5.2b)$$

$$\sum_{k+r+s+l=n+m} \delta_l \{ (k+r) g_r g_s'' - (k+s-l) g_r' g_s' \} = \text{Constant.} \quad (5.2c)$$

These constants are related to pressure gradient.

5.2 SOLUTION FOR THE CORE REGION :

In the core we take $m = 1$. Since the stream function is to match the supplied initial condition at $X = 0$, k is specified zero. Stream function is written as

$$\psi = \sum_n X^n F_n(\bar{Y}) \quad (5.3)$$

where $\bar{Y} = Y / H(X)$, $H(X) = \sum_n H_n X^n$, $H_0 = a$.

On substitution (5.2) reduces to

$$F_n''' + \sum_{r+s+l=n+1} H_l \{ r F_r F_s'' - (s-l) F_r' F_s' \} = D_n \quad (5.4)$$

where D_n is constant given by

$$H^3(X) P_X(X) = \sum_n X^n D_n \quad (5.5)$$

$D_n = 0$, $F_n = 0$ for $n < 0$.

The lowest order equation is obtained for n + 1 = 0

$$\sum_{r+s+l=0} H_l \left\{ r F_r F_s'' - (s-l) F_r' F_s' \right\} = 0$$

and is identically satisfied for any F_0. The initial condition requires that

$$F_0 = (3\,\overline{Y} - \overline{Y}^3) / 2 \tag{5.6a}$$

so that $F_0' = (3/2) (1 - \overline{Y}^2)$ (5.6b)

A general form of the equation (5.4) governing F_n can conveniently be written down as

$$n (F_0' F_n' - F_0'' F_n) = n \overline{H}_n (F_0')^2 - (D_{n-1} - F_{n-1}''') / a$$

$$+ \sum_{r<n} r \left(F_r F_{n-r}'' - F_r' F_{n-r}' \right) + \sum_{\substack{l=1 \\ r+s+l=n}}^{n-1} \overline{H}_l \left\{ r F_r F_s'' - (s-l) F_r' F_s' \right\} \tag{5.7}$$

where $\overline{H}_n = H_n / H_0 = H_n / a$

Equation (5.7) is singular as F_0', the coefficient of F_n' vanishes at $\overline{Y} = \pm 1$. Also because of the assumed symmetry the complimentary solution will be zero and only particular integral will contribute to the solution.

For n = 1 equation (5.7) is

$$(F_0' F_1' - F_0'' F_1) = \overline{H}_1 (F_0')^2 - (D_0 - F_0''') / a \tag{5.8a}$$

Its solution is

$$F_1 = (3/2)\,(1-\overline{Y}^2) \Big[\overline{H}_1 \overline{Y} + (4\overline{H}_1/9) \left\{ (1/4) \operatorname{Log} (1+\overline{Y}/1-\overline{Y}) + \overline{Y}/2(1-\overline{Y}^2) \right\} \Big] \tag{5.8b}$$

where $E_1 = (F_0''' - D_0) / a$ is constant.

$$F_1' = (3\bar{H}_1/2)(1-3\bar{Y}^2) + (E_1/3)\left\{ -\bar{Y} \operatorname{Log}(1+\bar{Y}/1-\bar{Y}) + 2 \right\} \quad (5.8c)$$

The above form of F_1' indicates a log singularity at $\bar{Y} = \pm 1$. This singularity can be attributed to the fact that at the edges of the core the condition that velocity is zero (before the sudden expansion) has been relaxed and consequently the local behaviour is different than that of elsewhere in the core. Hence a different kind of expansion is needed to explore this region, namely the shear layer. It is anticipated that the singularity which shall arise will be mild in nature.

5.3 SOLUTION FOR THE SHEAR LAYER :

The shear layer can be taken as a special case of weak mixing layer where a vortical stream comes in contact with fluid at rest. Based on earlier investigations (Chapter 3) we choose in equation (5.1), m = 3 and k = 2 and write the independent variables

$$\xi = (mX)^{1/m}, \qquad \eta = [Y - H(\xi)]/h_0\xi \quad (5.9a)$$

where $H(\xi) = H(X)$ and the stretching function $h(\xi)$ is simply ξh_0. The form of the stream function is choosen as

$$\psi = \sum_n \xi^{2+n} \left[\log\xi \cdot g_n(\eta) + f_n(\eta)\right] \quad (5.9b)$$

Equations (5.2) reduce to

$$f_n''' + h_0 \sum_{r+s=n} \left[f_s'' \{(r+2) f_r + g_r\} - f_r' \{(s+1) f_s' + g_s'\} \right] = \text{constant} \quad (5.10a)$$

$$g_n''' + h_0 \sum_{r+s=n} \Big[g_s'' \{(r+2) f_r + g_r\} + (r+2) g_r g_s'' - g_r' \{(s+1) f_s' + g_s'\} - (s+1) f_r' g_s' \Big] = \text{Constant} \quad (5.10b)$$

$$h_0 \sum_{r+s=n} \Big[(r+2) g_r g_s'' - (s+1) g_r' g_s' \Big] = \text{Constant} \quad (5.10c)$$

The left hand side of (5.10) should be independent of η and these constants are related to pressure gradient.

Though a general form of the governing equations is written the procedure is that g_n is taken to be zero unless need arises.

In the lowest order equation, the choice of m has led to the appearance of inertia and diffusion terms together. The lowest order equation is (g_0 is taken zero)

$$f_0''' + h_0 (2 f_0 f_0'' - f_0' f_0') = \text{constant} \quad (5.11)$$

This equation has been well investigated in Chapter 3. For large η, diffusion term drops to zero faster than the convective term. Also f behaves as a quadratic in η plus an exponentially small term (EST) for large $|\eta|$ towards the vertical stream and as a constant plus EST towards the fluid at rest. This can readily be seen as on substitution $f_0 \sim \eta^n$ in (5.11) one obtains n = 2 or 0. Writing

$$f_0 \sim a_1 \eta^2 + a_2 \eta + a_3 + \text{EST}, \text{ for large } |\eta| \quad (5.12)$$

the left hand side of (5.11) is $h_0(4a_1a_3 - a_2^2)$ which is constant and hence there is no restriction on a_1, a_2 and a_3.

The next order governing equation is (g_1 is taken zero)

$$f_1''' + h_0(3f_0''f_1 + 2f_0f_1'' - 3f_0'f_1') = \text{constant} \qquad (5.13)$$

Taking $f_0 \sim \eta^2$ and $f_1 \sim \eta^n$ for large $|\eta|$ we find that $n = 1$ or 3. Let

$$f_1 \sim a_4\eta^3 + a_5\eta^2 + a_6\eta + a_7 + EST \qquad (5.14)$$

for large $|\eta|$. For (5.13) to be independent of η we get either ($a_2^2 - 4a_1a_3$) = 0 or $a_4 = a_5 = 0$. We assume the first one. If it is not true then left hand side of (5.11) is nonzero and hence the leading term in the expansion of pressure gradient will be $O(\xi^{-1})$. We get therefore from (5.13)

$$a_5 = (3a_2/2a_1) a_4 \qquad (5.15a)$$

with additional condition

$$6a_4 + h_0 (4a_3a_5 - 3a_2a_6 + 6a_1a_7) = \text{constant} \qquad (5.15b)$$

While carrying out a similar analysis for the mixing layer for large η towards the fluid at rest we obtain

$$f_0 \sim b_1 + EST \qquad (5.16a)$$

$$f_1 \sim b_2\eta^2 + b_3\eta + b_4 + EST \qquad (5.16b)$$

Then the left hand side of (5.11) and (5.13) shall be zero and $4h_0b_1b_2$ with no additional condition on constants b_i s.

Higher order equations can be analysed in a similar manner. We shall return to it whenever the need arises.

5.4 MATCHING OF CORE AND SHEAR LAYER EXPANSIONS :

A general matching procedure (Kaplun 1967) is outlined below.

The variables in core are X, $\bar{Y} = Y/H(X)$ and the stream function expansion $\psi = \sum_n X^n F_n(\bar{Y})$. The expansion for velocity valid in core region is

$$\mathcal{C}(U) = \sum_n X^n F_n' / H(X)$$

$$= F_0' + X (F_1' - \bar{H}_1 F_0') + 0(X^2) \qquad (5.17)$$

In shear layer the variables are $\xi = (3X)^{1/3}$ and $\eta = (Y-H)/h_0\xi$. The stream function expansion is $\psi = \sum_n \xi^{2+n} [\log\xi \, g_n + f_n]$. Therefore the expansion for velocity valid in shear layer is

$$\mathcal{S}(U) = \sum_n \xi^{2+n} [\log\xi \, g_n' + f_n'] / h_0\xi \qquad (5.18)$$

Introduce the intermediate variable

$$Y_\mu = [Y - H(X)] / \mu(X) \qquad (5.19)$$

where $H(X) \supset \mu(X) \supset h(X)$. That is, $h/\mu \to 0$, $\mu/H \to 0$, μ is an intermediate length scale (it does not represent coefficient of viscosity). Therefore

$$\eta = Y_\mu (\mu / h_0 \varepsilon)$$
$$\bar{Y} = 1 + Y_\mu (\mu / H)$$

Then the matching condition is written as

$$\lim_{\mu} [\mathcal{C}(U) - \mathcal{S}(U)] / \mu_i(X) = 0 \tag{5.20}$$

In the above limit, $X \rightarrow 0$ for fixed Y_μ. The matching is said to be done to the order of the gauge function μ_i (X).

Consider the matching

$$\lim_{\mu} [\mathcal{C}(U) - \mathcal{S}(U)] / \varepsilon = 0 \tag{5.21}$$
$$F_0'(\bar{Y}) = (3/2)(1 - \bar{Y}^2)$$
$$= (-3/2)[2 Y_\mu \mu/H + Y_\mu^2 (\mu/H)^2]$$
$$f_0'(\eta) = f_0'(Y_\mu \mu / h_0 \varepsilon)$$
$$\approx 2a_1 Y_\mu \mu / h_0 \varepsilon + a_2 \text{ for large } \eta$$

The matching leads to $a_1 = -3h_0^2/2a$ and $a_2 = 0$.

Therefore $a_3 = 0$, $a_5 = 0$. Now consider the matching

$$\lim_{\mu} [\mathcal{C}(U) - \mathcal{S}(U)] / \varepsilon^2 = 0 \tag{5.22}$$
$$f_1'(\eta) = f_1'(Y_\mu \mu / h_0 \varepsilon)$$
$$\sim 3a_4 Y_\mu^2 (\mu / h_0 \varepsilon)^2 + a_6 \text{ for large } \eta$$

and matching leads to $a_4 = -h_0^3/2a^3$ and $a_6 = 0$

For the next order matching a gauge function $\varepsilon^3 \log \varepsilon$ is taken, because F_1' has its leading term as log. In the shear layer expansion log gauge functions start appearing which means that g_2 and so on are not necessarily zero. Before the matching is considered we like to see the behaviour of g_2. The equation governing g_2 is

$$g_2''' + h_0 \left(2g_2''f_0 + 4g_2 f_0'' - 4g_2' f_0'\right) = \text{constant} \tag{5.23}$$

Therefore $g_2 \sim \eta$. Let

$$g_2 \sim e_1 \eta + e_2 \qquad \text{for large } |\eta| \tag{5.24}$$

and then left hand side of (5.23) is like $8h_0 e_2 a_1$ hence there is no restriction on the constants e_i s. From the matching

$$\lim_{\mu} \left[G(U) - S(U)\right] / \varepsilon^3 \log \varepsilon = 0 \tag{5.25}$$

we get that if $\log(\mu/\varepsilon) / \log \varepsilon \to 0$, $e_1 = E_1 h_0 / 9a$. This restriction on μ means that there is a narrow overlap (e.g. $\mu \sim \varepsilon \log \varepsilon$). Otherwise the matching can be accomplished only when e_1 and E_1 are zero.

5.5 BUBBLE :

Suppose that the stream function in bubble behaves like $X^k \tilde{F}_0(\tilde{Y})$ where $\tilde{Y} = (Y-1)/(1-H)$. The diffusion term dominate if $k > 1$ which is not the case as ψ is like $X^{2/3}$ in shear layer. For $k < 1$ inertia terms dominate.

Therefore taking inertia terms only the governing equation

$$\tilde{F}_0 \tilde{F}_0'' - \tilde{F}_0'^2 = 0$$

gives a solution which on satisfying the condition $\tilde{F}_0'(0) = 0$ becomes a constant. Further the condition $\tilde{F}_0(0) = 0$ would require it to vanish identically. Therefore it seems unlikely that the stream function (or normal velocity) can be matched, though it was thought that such a matching will be needed as U goes to zero exponentially in the shear layer. It can therefore be thought that there is a layer close to the wall $X = 0$, called wall layer, which supplies the entrainment corresponding to the leading term. The wall is considered a singular surface as our assumption (that the length scale in the direction of channel behaves like R) definitely will not be valid there. Any arbitrary surface (to the order 1) will be reduced to a plane surface in the limit $R \to \infty$.

Hopefully this difficulty can be bypassed by matching U. We therefore retain the viscous term also.

$$\tilde{F}_0''' + (1-a)\,(\tilde{F}_0 \tilde{F}_0'' - \tilde{F}_0' \tilde{F}_0') = \tilde{A}_0$$

By redefining the independent variable and the constant $\tilde{A}_0$ we write the above equation as

$$\tilde{F}_0''' + \tilde{F}_0 \tilde{F}_0'' - \tilde{F}_0' \tilde{F}_0' = \tilde{A}_0$$

To obtain solution we write

$$\tilde{F}_0'''' + \tilde{F}_0 \tilde{F}_0''' - \tilde{F}_0' \tilde{F}_0'' = 0$$

The boundary conditions are $\tilde{F}_0(0) = \tilde{F}'_0(0) = 0,\ \tilde{F}'''_0 = \tilde{A}_0$.

Assuming a series solution

$$\tilde{F}_0 = \sum_n \Gamma_n \tilde{Y}^n$$

the recursive relation for the coefficients Γ_n are

$$\Gamma_{n+4} = \sum_{r=0}^{n} \Big\{ \Gamma_{r+1} \Gamma_{n-r+2} (r+1)(n-r+2)(n-r+1) - \Gamma_r \Gamma_{n-r+3} (n-r+3)(n-r+2)(n-r+1) \Big\} \Big/ \big[(n+4)(n+3)(n+2)(n+1)\big]$$

where $\Gamma_0 = \Gamma_1 = 0$ & $\Gamma_3 = \tilde{A}_0/6$. The unknown Γ_2 is to be found by matching with the shear layer.

In these sections a method of constructing solution for small X has been outlined and some steps have been carried out.

6. CONCLUSION

A representative problem of the class internal separated flow has been investigated for large Reynolds numbers using an asymptotic method. The problem choosen is the flow in a sudden expansion in a channel. It is assumed that the length scales are O(R) and O(1) in stream-wise and transverse direction. This is compatible with the computation made by Hung and Macagno (1966). Later Macagno and Hung (1967) attempted numerically and experimentally flow past a conduit expansion. Their findings once again support the basic assumption.

The main consequence of the assumption is that the Navier-Stokes equations become in this limit boundary layer equations. The problem differs in two ways from a classical boundary layer problem. First, there are four boundary conditions instead of three. Second, pressure distribution is not known and its determination is a part of the problem.

The method of solution is to construct coordinate expansions for small X and for large X. For the expansions of the first type, one needs three limit expansions valid in core, mixing layer and the bubble.

The matching procedure has been outlined. Intitutively the concept of matching in problems of parameter purturbation

has been extended. Such a matching had to be devised to render the solutions mutually compatible. Initial steps of matching has been shown and no trouble is encountered. This leads one to suspect that what has been taken as an assumption is realistic.

Flow at large distances from the sudden expansion (Chapter 4) leads to an eigenvalue problem and the determined eigenvalues are in excellent agreement with that obtained by Gillis and Brandt (1964) and Wilson (1969); they have used the complete Navier-Stokes equation. This means that the limit considered does bring out the essential features of the flow.

To complete the work, one would need additional terms in the inner expansion, patching the expansions at small and large X, and comparison with the results of Macagno and Hung. Definitive judgment about the validity of the expansions and corresponding limits can be made only afterwards. At the present stage, no major feature is revealed which would be difficult to explain in terms of qualitative physical arguments and this is an indication, but not a demonstration that the essential aspects of arguments are valid.

The study of initial development of the mixing layer has led to the identification of a family of viscous layers across which sharp changes in vorticity occur unaccompanied by sharp changes in velocity. A complete study of these

layers reveal several interesting properties. For example, its thickness varies like $X^{1/3}$ unlike the strong mixing layer which grows like $X^{1/2}$.

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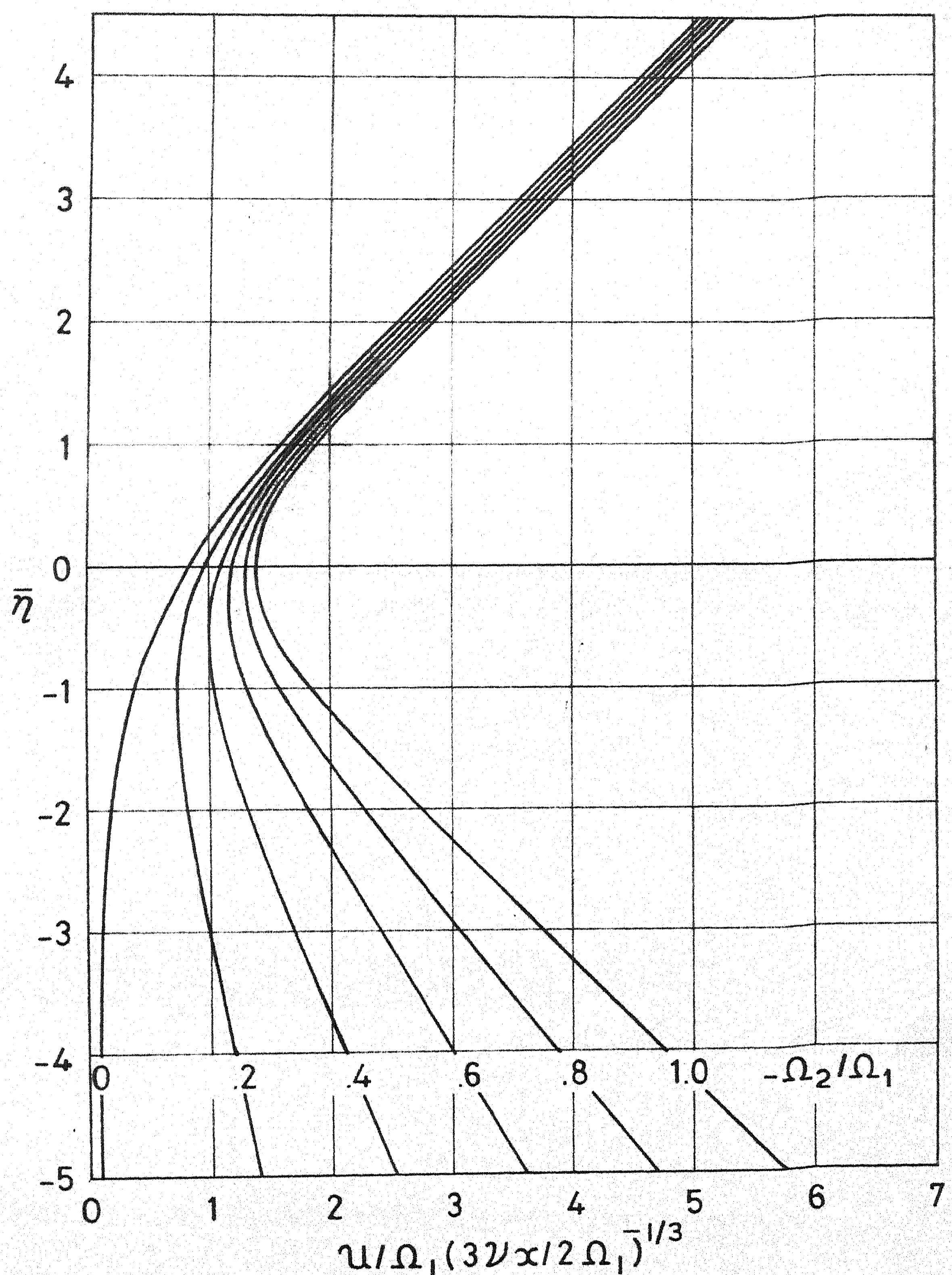

Fig. 1 – Velocity profile of weak mixing layer

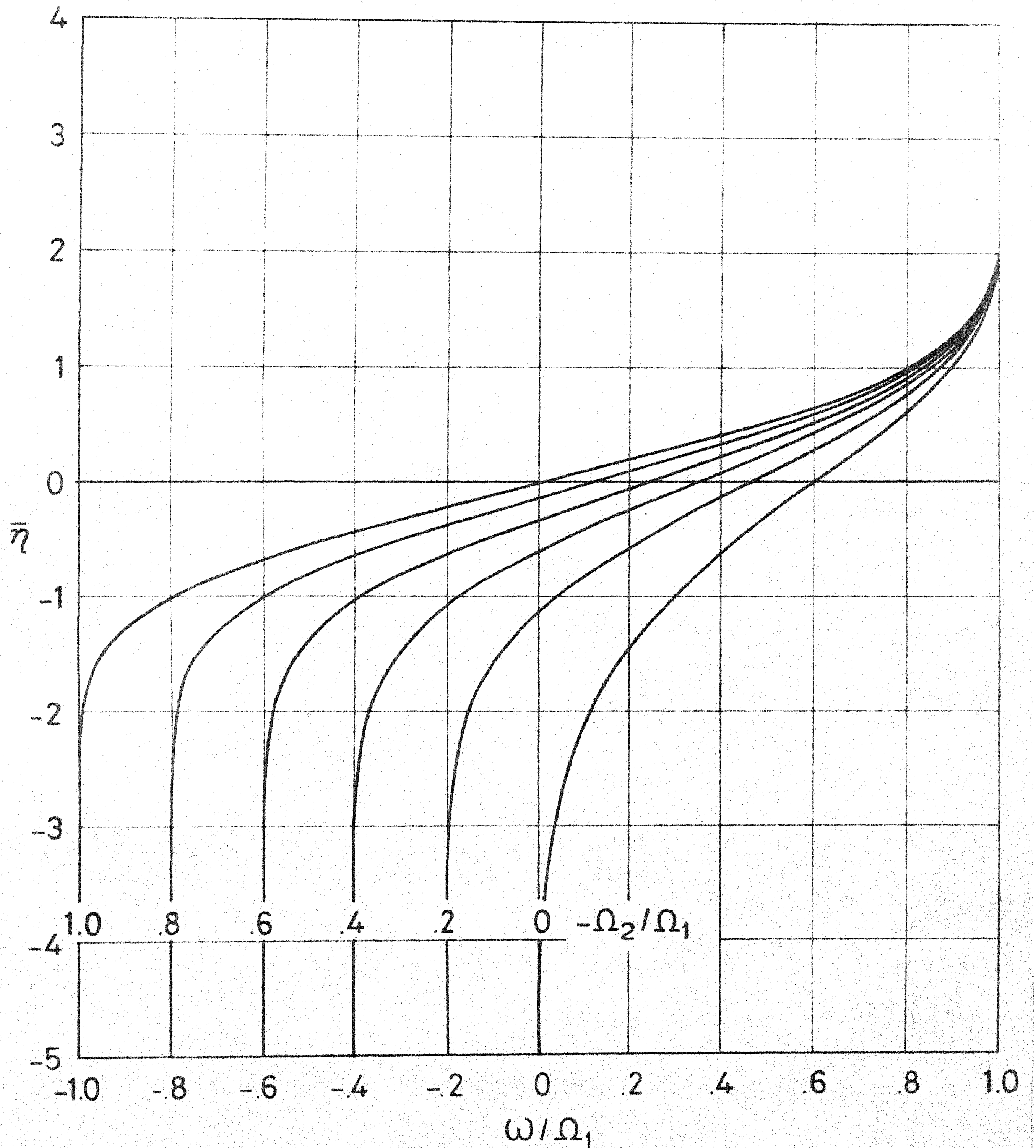

Fig. 2 – Vorticity profile of weak mixing layer

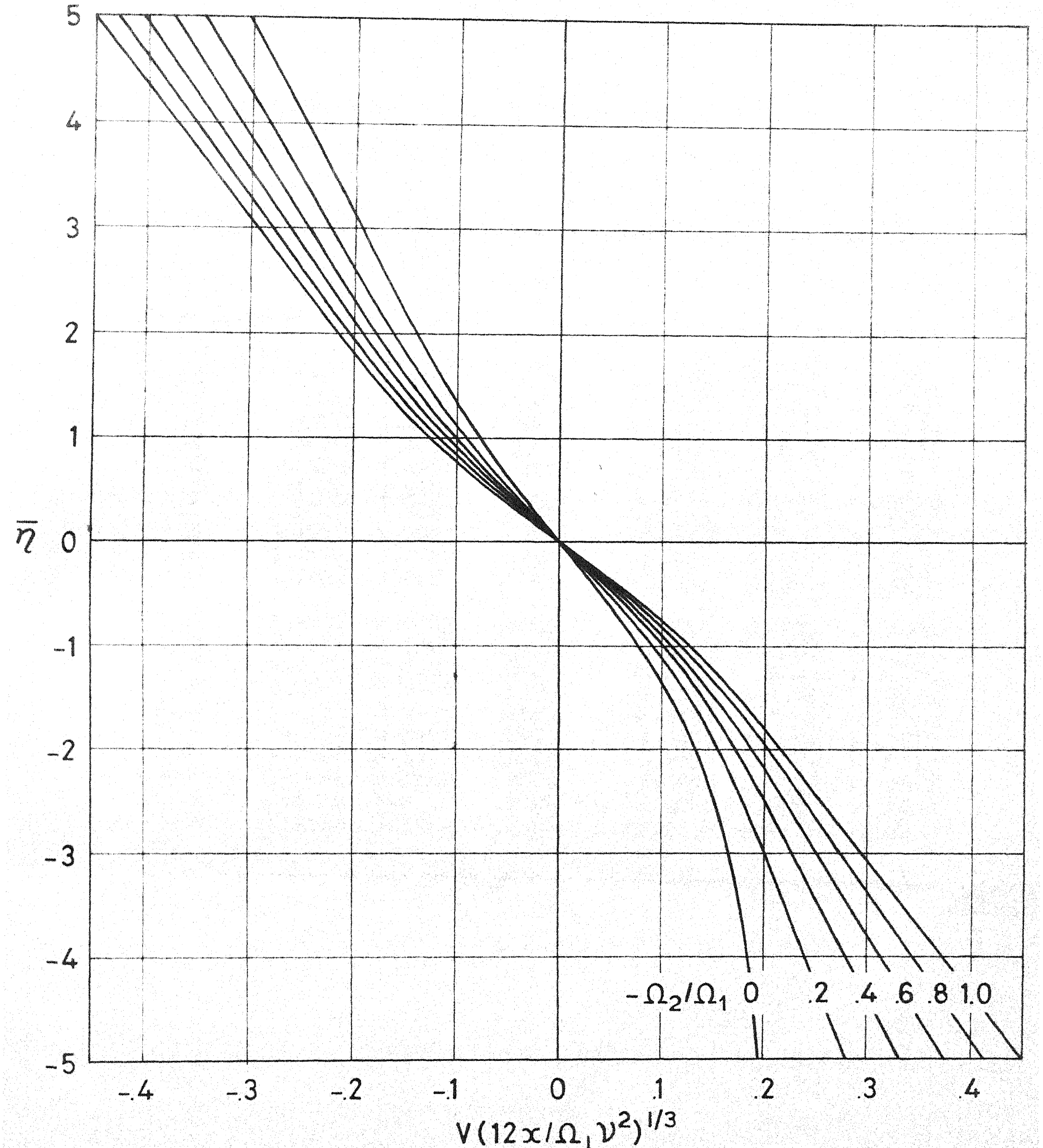

Fig. 3 - Normal velocity profile of weak mixing layer

TABLE 1A

Distribution of Velocity and Vorticity in

Weak Mixing Layer for $\Omega_2 / \Omega_1 = -1.0$

$\lvert\bar{\eta}\rvert$	$\bar{\eta}>0$ f	f'	f''	$\bar{\eta}<0$ f	f'	f''
0	0	1.4073	0	0	1.4073	0
.1	.1409	1.4122	.0988	- .1409	1.4122	- .0988
.2	.2828	1.4270	.1962	- .2828	1.4270	- .1962
.3	.4266	1.4514	.2909	- .4266	1.4514	- .2909
.4	.5734	1.4850	.3816	- .5734	1.4850	- .3816
.5	.7239	1.5275	.4673	- .7239	1.5275	- .4673
.6	.8791	1.5783	.5470	- .8791	1.5783	- .5470
.7	1.0398	1.6367	.6200	-1.0398	1.6367	- .6200
.8	1.2067	1.7021	.6858	-1.2067	1.7021	- .6858
.9	1.3804	1.7736	.7442	-1.3804	1.7736	- .7442
1.0	1.5616	1.8506	.7950	-1.5616	1.8506	- .7950
1.2	1.9482	2.0181	.8749	-1.9482	2.0181	- .8749
1.4	2.3697	2.1989	.9292	-2.3697	2.1989	- .9292
1.6	2.8283	2.3884	.9631	-2.8283	2.3884	- .9631
1.8	3.3254	2.5832	.9824	-3.3254	2.5832	- .9824
2.0	3.8618	2.7808	.9924	-3.8618	2.7808	- .9924
2.2	4.4378	2.9798	.9971	-4.4378	2.9798	- .9971
2.4	5.0537	3.1794	.9990	-5.0537	3.1794	- .9990
2.6	5.7096	3.3793	.9997	-5.7096	3.3793	- .9997
2.8	6.4054	3.5792	.9999	-6.4055	3.5793	- .9999
3.0	7.1413	3.7792	1.0000	-7.1413	3.7792	-1.0000

TABLE 1B

Distribution of Velocity and Vorticity in

Weak Mixing Layer for $\Omega_2 / \Omega_1 = -.8$

		$\bar\eta > 0$			$\bar\eta < 0$	
$\lvert\bar\eta\rvert$	$\bar f$	$\bar f'$	$\bar f''$	$\bar f$	$\bar f'$	$\bar f''$
0	0	1.3113	.1139	0	1.3113	.1139
.1	.1318	1.3270	.1997	- .1307	1.3042	.0281
.2	.2657	1.3512	.2844	- .2611	1.3056	-.0565
.3	.4024	1.3838	.3668	- .3921	1.3154	-.1390
.4	.5427	1.4244	.4460	- .5245	1.3333	-.2184
.5	.6875	1.4728	.5210	- .6592	1.3590	-.2936
.6	.8375	1.5285	.5910	- .7965	1.3919	-.3641
.7	.9934	1.5908	.6554	- .9376	1.4316	-.4292
.8	1.1559	1.6594	.7138	-1.0830	1.4775	-.4885
.9	1.3255	1.7334	.7657	-1.2333	1.5291	-.5416
1.0	1.5027	1.8123	.8112	-1.3890	1.5857	-.5887
1.2	1.8819	1.9822	.8835	-1.7185	1.7114	-.6646
1.4	2.2964	2.1642	.9332	-2.0744	1.8500	-.7186
1.6	2.7481	2.3542	.9647	-2.4591	1.9976	-.7543
1.8	3.2384	2.5492	.9829	-2.8739	2.1509	-.7762
2.0	3.7680	2.7468	.9925	-3.3196	2.3075	-.7886
2.2	4.3372	2.9458	.9970	-3.7970	2.4659	-.7950
2.4	4.9463	3.1455	.9990	-4.3061	2.6252	-.7980
2.6	5.5954	3.3453	.9997	-4.8471	2.7850	-.7993
2.8	6.2845	3.5453	.9999	-5.4200	2.9449	-.7998
3.0	7.0135	3.7453	1.0000	-6.0250	3.1049	-.7999
3.2				-6.6620	3.2648	-.8000

TABLE 1C

Distribution of Velocity and Vorticity in

Weak Mixing Layer for $\Omega_2 / \Omega_1 = -.6$

$\lvert\bar{\eta}\rvert$	$\bar{\eta} > 0$ $\bar{f}$	$\bar{f}'$	$\bar{f}''$	$\bar{\eta} < 0$ $\bar{f}$	$\bar{f}'$	$\bar{f}''$
0	0	1.2102	.2281	0	1.2102	.2281
.1	.1223	1.2367	.3012	-.1200	1.1911	.1550
.2	.2476	1.2704	.3734	-.2385	1.1792	.0828
.3	.3766	1.3113	.4438	-.3561	1.1745	.0123
.4	.5101	1.3591	.5116	-.4736	1.1767	-.0558
.5	.6487	1.4135	.5761	-.5916	1.1855	-.1207
.6	.7930	1.4742	.6365	-.7109	1.2007	-.1819
.7	.9437	1.5407	.6923	-.8320	1.2218	-.2389
.8	1.1013	1.6125	.7431	-.9554	1.2483	-.2913
.9	1.2663	1.6891	.7886	-1.0818	1.2798	-.3390
1.0	1.4393	1.7700	.8287	-1.2115	1.3159	-.3817
1.2	1.8103	1.9425	.8929	-1.4829	1.3997	-.4528
1.4	2.2170	2.1259	.9377	-1.7722	1.4958	-.5057
1.6	2.6611	2.3165	.9665	-2.0818	1.6009	-.5428
1.8	3.1439	2.5117	.9835	-2.4130	1.7121	-.5673
2.0	3.6660	2.7094	.9926	-2.7669	1.8272	-.5825
2.2	4.2277	2.9084	.9970	-3.1441	1.9447	-.5913
2.4	4.8294	3.1081	.9989	-3.5448	2.0634	-.5960
2.6	5.4710	3.3079	.9997	-3.9695	2.1829	-.5983
2.8	6.1525	3.5079	.9999	-4.4180	2.3228	-.5993
3.0	6.8741	3.7079	1.0000	-4.8905	2.4226	-.5998
3.2				-5.3871	2.5426	-.5999
3.4				-5.9076	2.6625	-.6000

TABLE 1D

Distribution of Velocity and Vorticity in

Weak Mixing Layer for $\Omega_2 / \Omega_1 = -.4$

$\lvert\eta\rvert$	$\eta > 0$ f	f'	f''	$\eta < 0$ f	f'	f''
0	0	1.1024	.3432	0	1.1024	.3432
.1	.1121	1.1397	.4038	- .1086	1.0711	.2825
.2	.2281	1.1831	.4638	- .2144	1.0459	.2225
.3	.3489	1.2324	.5224	- .3180	1.0265	.1638
.4	.4748	1.2875	.5790	- .4199	1.0130	.1069
.5	.6066	1.3482	.6330	- .5208	1.0051	.0523
.6	.7446	1.4140	.6838	- .6211	1.0025	.0006
.7	.8896	1.4848	.7310	- .7214	1.0049	-.0481
.8	1.0417	1.5601	.7741	- .8222	1.0120	-.0934
.9	1.2017	1.6395	.8130	- .9240	1.0234	-.1351
1.0	1.3698	1.7225	.8476	-1.0271	1.0389	-.1731
1.2	1.7316	1.8979	.9033	-1.2388	1.0802	-.2381
1.4	2.1296	2.0828	.9429	-1.4599	1.1332	-.2889
1.6	2.5652	2.2742	.9688	-1.6926	1.1949	-.3268
1.8	3.0396	2.4696	.9843	-1.9383	1.2632	-.3539
2.0	3.5532	2.6674	.9929	-2.1982	1.3359	-.3723
2.2	4.1065	2.8665	.9971	-2.4729	1.4117	-.3842
2.4	4.6998	3.0661	.9989	-2.7630	1.4893	-.3914
2.6	5.3330	3.2660	.9997	-3.0687	1.5680	-.3956
2.8	6.0062	3.4660	.9999	-3.3902	1.6474	-.3979
3.0	6.7193	3.6660	1.0000	-3.7277	1.7271	-.3990
3.2				-4.0811	1.8070	-.3996
3.4				-4.4505	1.8869	-.3998
3.6				-4.8358	1.9669	-.3999
3.8				-5.2372	2.0469	-.4000

TABLE 1E

Distribution of Velocity and Vorticity in

Weak Mixing Layer for $\Omega_2/\Omega_1 = -.2$

	$\bar{\eta} > 0$			$\bar{\eta} < 0$		
$\lvert\bar{\eta}\rvert$	$\bar{f}$	$\bar{f}'$	$\bar{f}''$	$\bar{f}$	$\bar{f}'$	$\bar{f}''$
0	0	.9838	.4613	0	.9838	.4613
.1	.1008	1.0323	.5096	- .0961	.9400	.4130
.2	.2066	1.0857	.5574	- .1882	.9011	.3651
.3	.3181	1.1437	.6042	- .2765	.8670	.3181
.4	.4355	1.2065	.6496	- .3617	.8375	.2725
.5	.5595	1.2736	.6931	- .4442	.8125	.2284
.6	.6904	1.3450	.7341	- .5224	.7917	.1863
.7	.8286	1.4203	.7725	- .6027	.7751	.1461
.8	.9746	1.4994	.8078	- .6795	.7624	.1086
.9	1.1286	1.5818	.8398	- .7553	.7553	.0734
1.0	1.2910	1.6672	.8684	- .8303	.7476	.0408
1.2	1.6422	1.8458	.9152	- .9794	.7454	-.0167
1.4	2.0299	2.0324	.9490	-1.1291	.7536	-.0640
1.6	2.4555	2.2247	.9716	-1.2814	.7703	-.1017
1.8	2.9200	2.4205	.9854	-1.4377	.7937	-.1309
2.0	3.4239	2.6184	.9932	-1.5993	.8222	-.1528
2.2	3.9674	2.8175	.9971	-1.7669	.8546	-.1688
2.4	4.5509	3.0171	.9989	-1.9412	.8894	-.1800
2.6	5.1743	3.2170	.9996	-2.1228	.9262	-.1876
2.8	5.8377	3.4170	.9999	-2.3118	.9643	-.1926
3.0	6.5411	3.6170	1.0000	-2.5085	1.0031	-.1957
3.2				-2.7131	1.0425	-.1976
3.4				-2.9255	1.0821	-.1987
3.6				-3.1459	1.1219	-.1993
3.8				-3.3743	1.1618	-.1997
4.0				-3.6107	1.2018	-.1999
4.2				-3.8550	1.2418	-.1999
4.4				-4.1074	1.2818	-.2000

TABLE 1F

Distribution of Velocity and Vorticity in

Weak Mixing Layer for $\Omega_2 / \Omega_1 = - .0$

$\lvert\bar{\eta}\rvert$	$\bar{f}$ ($\bar{\eta} > 0$)	$\bar{f}'$ ($\bar{\eta} > 0$)	$\bar{f}''$ ($\bar{\eta} > 0$)	$\bar{f}$ ($\bar{\eta} < 0$)	$\bar{f}'$ ($\bar{\eta} < 0$)	$\bar{f}''$ ($\bar{\eta} < 0$)
0	0	.8285	.6016	0	.8285	.6016
.1	.0859	.8903	.6359	- .0799	.7700	.5673
.2	.1782	.9556	.6699	- .1541	.7150	.5333
.3	.2771	1.0248	.7032	- .2230	.6633	.4999
.4	.3831	1.0962	.7357	- .2869	.6150	.4671
.5	.4965	1.1714	.7669	- .3461	.5699	.4353
.6	.6175	1.2496	.7967	- .4010	.5279	.4046
.7	.7465	1.3307	.8246	- .4518	.4889	.3750
.8	.8837	1.4144	.8505	- .4988	.4528	.3468
.9	1.0295	1.5007	.8743	- .5425	.4195	.3199
1.0	1.1840	1.5892	.8957	- .5828	.3888	.2945
1.2	1.5200	1.7721	.9314	- .6550	.3347	.2479
1.4	1.8932	1.9611	.9577	- .7173	.2892	.2071
1.6	2.3047	2.1546	.9758	- .7712	.2515	.1717
1.8	2.7552	2.3510	.9872	- .8183	.2202	.1413
2.0	3.2452	2.5492	.9938	- .8597	.1949	.1156
2.2	3.7750	2.7483	.9973	- .8965	.1737	.0938
2.4	4.3446	2.9480	.9989	- .9295	.1568	.0757
2.6	4.9541	3.1478	.9996	- .9594	.1433	.0606
2.8	5.6037	3.3478	.9999	- .9869	.1324	.0482
3.0	6.2933	3.5478	1.0000	-1.0125	.1238	.0380
3.2				-1.0366	.1171	.0296
3.4				-1.0595	.1119	.0228
3.6				-1.0814	.1079	.0174
3.8				-1.1027	.1049	.0129
4.0				-1.1234	.1026	.0094
4.2				-1.1438	.1011	.0066
4.4				-1.1639	.1000	.0043
4.6				-1.1838	.0993	.0025
4.8				-1.2036	.0989	.0011
5.0				-1.2234	.0988	.0000

TABLE 2A

Eigenvalues obtained from (4.11)

Even	Odd
14.4531	18.8136
48.8706	57.5226

TABLE 2B

Eigenvalues obtained for different Reynold numbers by Gillis and Brandt(1964)

R	λ
$R_{crit.} = 8.461$	22.25
10	19.10
25	18.73
50	18.79
100	18.81
250	18.81
Large R	18.81

TABLE 2C

$\bar{\lambda}_0$ in (4.12), obtained by Wilson(1969)

Even	Odd
14.45	18.81
48.87	57.52
104.43	

As $R \rightarrow \infty$, $\lambda = \bar{\lambda}_0 + O(R^{-2})$

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